THE GRAPHICAL SOLUTION OF SPHERICAL TRIANGLES.

BY HARRY C. BRADLEY, Massachusetts Institute of Technology.

Corresponding to every spherical triangle, there exists at the center of the sphere a trihedral angle, whose face angles are equal to the sides of the spherical triangle, and whose dihedral angles are equal to the angles of the triangle. Graphical solutions of the trihedral are readily obtained by descriptive geometry.

Gaspard Monge, the French genius who may be called the father of descriptive geometry, does not appear to have given us any solutions for the trihedral. At any rate, an edition of his descriptive geometry, dated 1820, the earliest edition to which I have had access, omits the subject of trihedrals entirely. The excellent work published by Prof. Albert E. Church of the U. S. Military Academy, West Point, in 1864, contains graphical solutions of the trihedral corresponding to all six cases which arise in the solution of spherical triangles, and is the earliest dated work which I have as yet discovered. A number of modern texts give these solutions. A list (far from complete) appears at the end of this article.[1]

---

[1] An early reference to the graphical solution of spherical triangles, similar to that of Professor Bradley's paper, is Chapter 17 ("Résolution des Triangles sphériques par la Regle et le Compas") in A. Cagnoli's *Traité de Trigonométrie rectiligne et sphérique* . . . traduit de l'Italien par M. Chompré. Paris, 1786; chapter 19 of the second edition, Paris 1808. (*Trigonometria piana e sferica.* Edizione seconda notabilmente ampliate. Bologna, 1804: "Risoluzioni de'triangoli sferici con la riga e col compasso," pp. 346–349). For similar and allied discussions the following sources may be consulted.

C. Gudermann, *Lehrbuch der niederen Sphärik,* Münster, 1836.

G. K. L. von Littrow, "Ueber Herrn M. Eble's graphische Methoden der Auflösung sphärischen Dreiecke mit besonderer Rücksicht auf sein neuestes 'Stundenzeiger' oder 'Horoskop' genanntes Instrument," *Sitzungsberichte der math.-naturwiss. Klasse d. k. Akademie der Wissen.,* Vienna, vol. 42, 1860, pp. 203–212.

F. C. Penrose, "Description of an improved diagram for the graphical solution of spherical triangles, applicable to the questions arising out of the spheroidal figure of the earth," *Monthly Notices of the Royal Astronomical Society,* vol. 37, May, 1877, pp. 403–409.

L. Janse, "Over het graphisch oplossen van bolvornigé driehoeken en van daarop gegronde zeevaart en sterrekundige vraagstukken," *Nieuw Archief voor Wiskunde,* vol. 11, 1884, pp. 1–27; vol. 12, 1886, pp. 113–148.

C. H. Smith, "A graphic method of solving spherical triangles," *Amer. Jl. Math.,* vol. 6, 1884, pp. 175–176.

C. Wiener, *Lehrbuch der darstellenden Geometrie,* vol. 1, Leipzig, 1884, pp. 104–113.

P. Braun, "Das Trigonometer von C. Braun," *Math. naturwissensch. Ber. aus Ungarn,* reprinted in W. Dyck, *Katalog mathematischer und mathematischphysikalischer Modelle, Apparate und Instrumente,* München, 1892, pp. 160–161.

M. D'Ocagne, (*a*) *Journal de l'Ecole Polytechnique,* second series, vol. 4, 1898, p. 224; (*b*) *Traité de Nomographie,* Paris, 1899, pp. 326–330.

S. Haller, "Beitrag zur Geschichte der konstruktiven Auflösung sphärischer Dreiecke durch stereographische Projektion," *Biblioteca Mathematica,* n.s., vol. 13, 1899, pp. 71–80.

F. N. Willson, *Theoretical and practical Graphics,* New York, 1902, pp. 206–210.

M. D'Ocagne, "Sur la résolution nomographique des triangles sphériques," (*a*) *Bull. Soc. Math. de France,* vol. 32, 1904, pp. 196–203; (*b*) *Comptes rendus de l'Académie des Sciences,* vol. 138, 1904, pp. 70–72.

G. Pesci, "Resolução nomographica do triangulo de Posição" [translated from the Italian into Portuguese by Radler de Aquino], *Revista Maritima Brazileira,* Nov.–Dec., 1907; Feb., 1908.

These graphical solutions, however, ordinarily appear merely as exercises in descriptive geometry. Without some modification or adaptation, they are not suited for general use as an aid in checking numerical computations of spherical triangles. Usually all parts of the triangle are taken as less than 90°. When several of the parts lie between 90° and 180°, the resulting construction often becomes very difficult, even with an expert knowledge of descriptive geometry, to execute and to interpret correctly. Yet, as a check to numerical computations, the graphical solutions are not without value. Especially is this true in those cases where ambiguity exists in the numerical solution. The graphical construction shows clearly one solution, two solutions, or none; and in the case of two solutions, the correspondence of the parts.

To test the accuracy which may be expected from a graphical solution, I drew a hundred or so figures of moderate size, say six or seven inches across. The angles were measured with a semi-circular protractor, five inches in diameter, and were laid out only to the nearest whole degree. Disregarding some extraordinary agreements, probably more or less accidental, an average accuracy of 1° or 2° was readily obtained. This is quite sufficient to detect any gross error of calculation.

Now, as an aid to checking numerical calculations, the fewer graphical solutions which can be made to serve, the better. Fortunately, direct graphical solutions of all possible cases of spherical triangles, with all possible combinations of acute and obtuse angles for given parts, are unnecessary, provided we are willing to combine a little trigonometry with our descriptive geometry. The trigonometry required is of the simplest sort, namely:

1. The principle of polar triangles, by which sides are replaced by the supplements of angles, and angles by the supplements of sides. For instance, the graphical solutions for three given sides are simple and direct, while those for three given angles are not. Hence, by applying the principle of polar triangles,

---

R. de Aquino, "Nomograms for deducing altitude and azimuth and for star identification and finding course and distance in great circle sailing," *U. S. Naval Institute Proceedings*, vol. 34, 1908, pp. 633–646. See also W. C. P. Muir, *Treatise on Navigation*, fourth edition, Annapolis, Md., 1918, Appendix D, pp. 773–777: "Solution of the astronomical triangle by nomography."

G. Pesci, "Cenni sulla risoluzione del triangolo di posizione senza calcoli trigonometrici," *Rivista Marittima*, Roma, vol. 42, Sept., 1909, pp. 317–328 + 1 table. [Discusses the "compasso trigonometrico" of G. F. Richer described by F. Callet in the supplement to Bézout's Spherical Trigonometry and Navigation, Paris, 1798].

G. Pesci, "Sulla risoluzioni dei triangoli sferici senza calcoli trigonometrici" *Supplemento al Periodico de Matematica*, Livorno, vol. 13, 1910, pp. 65–73.

C. Schoy, *Beiträge zur konstruktiven Lösung sphärischastronomischer Aufgaben*. Leipzig, 1910. 7 + 40 pp. + 8 plates.

G. Loria, *Vorlesungen über darstellende Geometrie*, translated by F. Schütte, part 2, Leipzig, 1913, pp. 3–15.

M. D'Ocagne, *Cours de Géométrie*, tome 2, Paris, 1918, pp. 312–314.

H. G. G., "A new graphic method in nautical astronomy," *Nature*, vol. 102, Oct. 24, 1918, pp. 155–6.

A. Hutchinson and H. B. Goodwin, "Graphic methods in astronomy," *Nature*, vol. 103, March 13, 20, 1919, pp. 25, 44. Graphic methods here employed are "recommended to crystallographers."

EDITOR.

both cases can be solved graphically by the construction for three given sides.

2. The principle of co-lunar triangles. If the given parts are of such magnitudes that the direct graphical solution is difficult of execution and interpretation, one, at least, of the three possible co-lunar triangles will give a simpler solution.

The problem which I set myself, then, was to find: 1. Which of the graphical solutions were simplest to construct and easiest to interpret; 2. The minimum number of such solutions necessary, after making any trigonometrical transformations in the data which seemed desirable. I submit the following constructions as my result. There is nothing especially new in them. Under the circumstances, there could not be. But one or two of them differ more or less in detail from anything I have ever happened to see published.

These constructions were all made by descriptive geometry. Those familiar with that subject will readily recognize the horizontal and vertical coördinate planes, the ground line, traces of oblique planes, and revolution about those traces. The constructions, however, may readily be made without knowledge of descriptive geometry, and to render them more generally useful will be described merely as constructions.[1] In each example, the angles of the given triangle are A, B, C, the opposite sides a, b, c. To make the examples more specific, actual numerical values (in degrees) for the parts of the triangle will be used throughout.

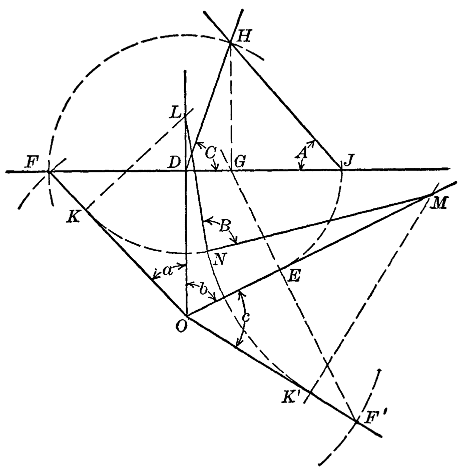

Fig. 1.

Case I. Given the three sides.

Example 1. All given sides acute. Given $a = 43°$, $b = 64°$, $c = 58°$.

This is solved in Fig. 1. Select any point O, and draw a vertical line OD. To the left of this line, lay off the smallest of the three given sides, in this case $a = 43°$. To the right of OD, lay off the largest side, $b = 64°$. Beyond this, lay off the remaining side, $c = 58°$. Draw a horizontal line intersecting the sides of a at F and D. Make $OF' = OF$. From F', draw $F'E$ perpendicular to OE, intersecting FD at G. At G, draw a vertical line. With D as center, radius DF, draw an arc intersecting this vertical at H. Draw HD. Then HDG is the angle $C = 70°$. With G as center, radius GE, draw an arc to intersect FD at J. Draw HJ. Then HJG is the angle $A = 49°$.

To find the remaining angle, opposite the middle side b, take any point K on OF. Draw from K a perpendicular to OF to meet OD at L. Make $OK' = OK$. From K' draw $K'M$ perpendicular to OF' to meet OE at M. With L as center,

[1] We have in this article a good illustration of the employment of the methods of descriptive geometry in pure mathematics as recommended very strongly by Professor Gino Loria in letters to Professor Roever published last year in this Monthly (*1919*, 45–53).—Editor.

radius LK, draw an arc. With M as center, radius MK', draw an arc, meeting the first arc at N. Then angle LNM is the angle $B = 85°$.

Answer. $A = 49°$, $B = 85°$, $C = 70°$.

Example 2. Two given sides acute, one obtuse. Given $a = 60°$, $b = 126°$, $c = 76°$.

The construction is shown in Fig. 2. It differs from Fig. 1 only in the size of the given parts. The figure is lettered the same as Fig. 1, and the same directions will apply.

Answer. $A = 35°$, $B = 148°$, $C = 40°$.

Example 3. Two given sides obtuse, one acute. Given $a = 60°$, $b = 100°$, $c = 140°$.

Form the co-lunar triangle $a = 60°$, $b' = 180° - b = 80°$, $c' = 180° - c = 40°$. Solve this by the construction of Fig. 1. This gives $A = 55°$, $B' = 112°$, $C' = 37°$. Whence, taking the co-lunar triangle of this result, we have the

Answer. $A = 55°$, $B = 68°$, $C = 143°$.

Example 4. All given sides obtuse. Given $a = 100°$, $b = 110°$, $c = 135°$.

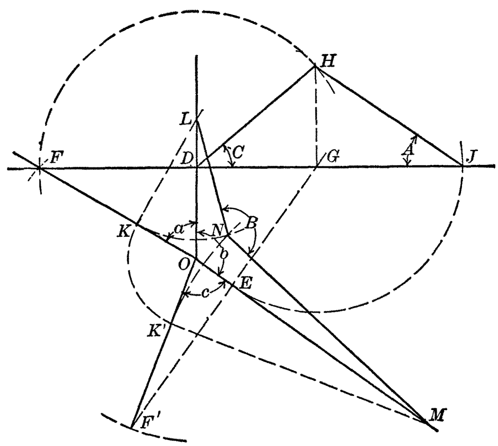

FIG. 2.

Form the co-lunar triangle $a = 100°$, $b' = 70°$, $c' = 45°$. Solve this by the construction of Fig. 2, giving $A = 129°$, $B' = 48°$, $C' = 34°$. Whence,

Answer. $A = 129°$, $B = 132°$, $C = 146°$.

Example 5. An impossible case. Given $a = 30°$, $b = 80°$, $c = 40°$.

If we overlook the fact that $a + c < b$, the construction will bring us to our senses. Proceeding as in Fig. 1, we find point G, and draw a vertical line at this point. But with the parts as given, DG will be greater than DF. Hence, when we take D as center, DF as radius, and draw an arc, the arc will not intersect the vertical line at G. Therefore there is no solution.

Case II. Given the three angles.

As already stated, no direct constructions for this case are needed. Take the polar triangle corresponding to the given triangle, and solve by Case I.

Case III. Given two sides and the included angle.

Example 6. All given parts acute. Given $a = 43°$, $b = 64°$, $C = 70°$.

This is the same triangle solved from different given parts in Fig. 1, and the completed solution will be the same figure. To construct the figure from the above given parts, draw a vertical line OD. To the left of OD lay off the smaller of the given sides, $a = 43°$. To the right, lay off $b = 64°$. Draw the horizontal line FD. At D lay off the given angle C as shown. With D as center, radius

DF, draw an arc to intersect DH at H. Draw a vertical line HG, to meet FD produced at G. Draw GE perpendicular to OM. With O as center, radius OF, draw an arc to intersect GE produced at F'. Draw OF'. Then EOF' is the third side, $c = 58°$. The three sides now being known, the remaining angles, $A = 49°$ and $B = 85°$, are found as previously explained.

Answer. $c = 58°$, $A = 49°$, $B = 85°$.

Example 7. The two given sides acute, the given angle obtuse. Given $a = 61°$, $b = 53°$, $C = 135°$.

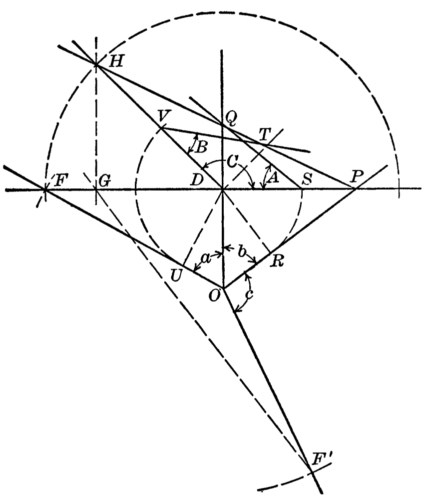

FIG. 3.

This example is solved in Fig. 3. Draw the vertical line OD. To the left of OD lay off the *larger* of the given sides, $a = 61°$. To the right of OD lay off the smaller given side, $b = 53°$. Draw a horizontal line FD, intersecting the sides of the angle a at F and D. Produce FD to intersect the remaining side of angle b at P; this can always be done, since the smaller side has been placed at the right of OD. At D, lay off the given angle $C = 135°$, as shown. With center D, radius DF, draw an arc intersecting DH at H. From H draw a vertical line to intersect FD at G. From G draw a line perpendicular to OP. With O as center, radius OF, draw an arc to intersect this line at F'. Then POF' is the third side, $c = 102°$.

Connect H with P, intersecting OD produced at Q. Draw DR perpendicular to OP. On DP, make distance $DS = DR$. Then DSQ is the angle opposite the side DOF, $A = 39°$.

From D, draw DT perpendicular to DH, intersecting HP at T. Also draw DU perpendicular to OF. Lay off on DH the distance $DV = DU$. Draw VT. Then DVT is the angle opposite the middle side, $B = 35°$.

Answer. $c = 102°$, $A = 39°$, $B = 35°$.

No other figures are needed for this Case. Any other conditions can be reduced to one of the two preceding solutions by the use of a co-lunar triangle. (See Examples 3 and 4.)

Case IV. Given two angles and the included side.

Take the polar triangle corresponding to the given triangle, and solve by Case III.

There are no impossible cases under Cases III and IV, and the construction can always be made.

Case V. Given two sides and the angle opposite to one of them.

Example 8. All three given parts acute. Given $a = 45°$, $b = 58°$, $A = 39°$.

The construction is shown in Fig. 4. Draw the vertical line OD. To the

left of OD lay off the side whose opposite angle is given, $a = 45°$. To the right of OD, lay off the other given side, $b = 58°$. Draw a horizontal line to give intersections F, D, and P. Draw DR perpendicular to OP. On DP, make $DS = DR$. At S, lay off the given angle A as shown, obtaining the intersection Q on OD produced. Draw PQ. With D as center, radius DF, draw an arc to intersect PQ. Since the side a is less than the side b, $DF < DP$, and the arc will intersect PQ in two points to the left of P, namely, H and H', each of which will give a solution to the problem.

For the first answer, from H draw a vertical line, intersecting DP at G. From G draw GE perpendicular to OP. With center O, radius OF, intersect this line at F'. Then EOF' is the third side, $c_1 = 18°$. Draw DH. Then GDH is the angle opposite this side, $C_1 = 16°$. The angle opposite the middle side, b, is found at N by the construction explained in Fig. 1; $B_1 = 131°$.

First Answer. $c_1 = 18°$, $C_1 = 16°$, $B_1 = 131°$.

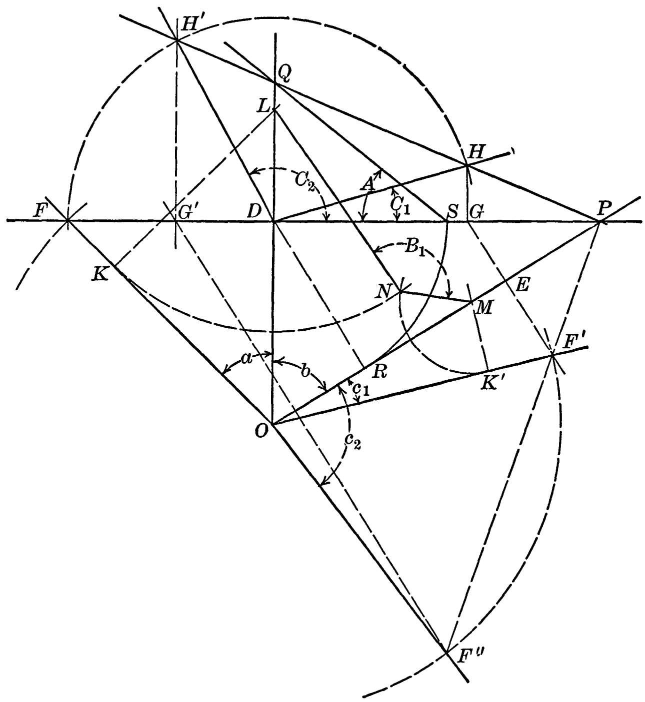

FIG. 4.

Proceeding similarly with the point H', we find EOF'' is $c_2 = 84°$, and GDH' is $C_2 = 118°$. No construction is needed for the remaining angle, B_2, since this angle is known by trigonometry to be the supplement of B_1. Hence $B_2 = 49°$.

Second Answer. $c_2 = 84°$, $C_2 = 118°$, $B_2 = 49°$.

Note. Should the intersection at F' prove rather flat, the point F' may be located more accurately from the fact that it lies on the line PF'', as shown in the figure.

Example 9. Given $a = 57°$, $b = 63°$, $A = 70°$.

If we proceed as explained for Fig. 4, the circle with D as center, radius DF, will be found to be tangent to PQ. There is then but one point H, and one solution. Going on with the construction from the point H, the third side is EOF', $c = 34°$. The angle opposite this side is GDH, $C = 38°$. The angle opposite the middle side needs no construction; by trigonometry, $B = 90°$.

Answer. $c = 34°$, $B = 90°$, $C = 38°$.

Example 10. Given $a = 38°$, $b = 59°$, $A = 53°$.

If we proceed as before, we shall find in this example that the radius DF is so short that the arc from F will not intersect PQ. There is consequently no solution.

Example 11. Given $a = 61°$, $b = 53°$, $A = 39°$.

This is the same triangle that was solved from different given parts in Fig. 3, and the completed construction will be the same figure.

The construction is started as explained for Fig. 4, Ex. 8, and proceeds as there explained until we draw the arc with D as center, radius DF. In this example, the side a, opposite the given angle A, is greater than the other given side b. Hence $DF < DP$, and the arc drawn from F will intersect the line PQ at but one point to the left of P. This intersection, H, gives the only solution to this example. Draw DH. Then PDH is the angle included between the given sides a and b, $C = 135°$.

Two sides and the included angle being known, the construction can now be completed as explained in case III. See Example 7.

Answer. $c = 102°$, $B = 35°$, $C = 135°$.

Example 12. Given $a = 63°$, $b = 58°$, $A = 75°$.

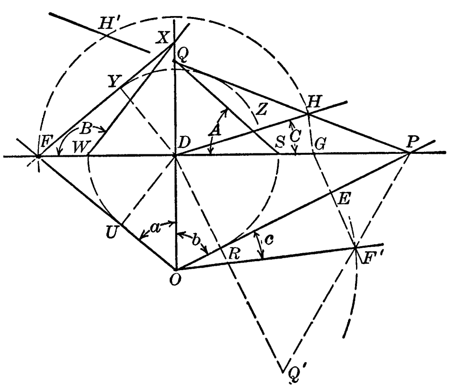

FIG. 5.

This is apparently the same case as that of the preceding example, since the side opposite the given angle is larger than the other. However, on attempting the construction, the given parts will be found to be of such sizes that the intersection H lies between P and Q. The angle included between the given sides is therefore acute, and the construction is best completed by the method used in Fig. 4 for the point H. See Example 8.

Answer. $c = 60°$, $B = 67°$, $C = 70°$.

Example 13. The two given sides acute, the given angle obtuse. Given $a = 51°$, $b = 64°$, $B = 129°$.

The construction is shown in Fig. 5. Draw the vertical line OD. At the *right* of this line, lay off the side whose opposite angle is given, $b = 64°$. Lay off the other side, $a = 51°$, to the left of OD. Draw a horizontal line to give the intersections F, D, and P. From D draw DU perpendicular to OF. Make the distance $DW = DU$. At W lay off the given angle, $B = 129°$, as shown, giving the intersection X on OD produced. Draw FX. From D draw DY perpendicular to FX. With D as center, radius DY, draw the arc YZ. From P draw the line PQ tangent to this arc, intersecting OD produced at Q.

With D as center, radius DF, draw an arc to intersect PQ. The intersection H', nearer F, does not give a solution to the problem. A solution exists only if a second intersection, H, can be found on PQ to the left of P. It is evident that this

can only occur when, as in the figure, $b > a$. If $b < a$, P will be inside the circle with radius DF, there will be no second intersection H, and no solution.

From H draw the vertical line HG, intersecting DP at G. From G draw GE perpendicular to OP. With O as center, radius OF, intersect this perpendicular at F'. Then EOF' is the third side, $c = 19°$. Draw HD. Then HDG is the angle opposite this side, $C = 16°$.

From D draw DR perpendicular to OP. Make the distance $DS = DR$. Draw SQ. Then DSQ is the remaining angle, $A = 42°$.

In case the intersection at F' should be too flat to be accurate, the point F' may be located as follows. Produce DR, which is perpendicular to OP. Make the distance $RQ' = SQ$. Draw $Q'P$; this line passes through F'.

Answer. $c = 19°$, $A = 42°$, $C = 16°$.

Any other example under this case can be reduced to one of those already given by the use of a co-lunar triangle.

Case VI. Given two angles and the side opposite to one of them.

Take the polar triangle corresponding to the given triangle, and solve by Case V.

Summary of the Constructions. In all the foregoing constructions, two of the given parts are always two sides, and acute. The constructions may then be summarized according to the third given part, as follows.

1. The third side, acute. Each side less than the sum of the other two. Fig. 1. One solution.

2. The third side, obtuse, but less than the sum of the other two. Fig. 2. One solution.

3. The third side, one of the three sides equal to, or greater than, the sum of the other two. No solution.

4. Angle included between the given sides, acute. Fig. 1. One solution.

5. Angle included between the given sides, obtuse. Fig. 3. One solution.

6. Angle opposite the smaller given side, acute. Fig. 4. Two solutions, one solution, or no solution.

7. Angle opposite the larger given side, acute. Start Fig. 4 and find the angle which is included between the given sides. If this angle is acute, complete Fig. 4. But if the included angle is obtuse, use the construction of Fig. 3. One solution.

8. Angle opposite the larger given side, obtuse. Fig. 5. One solution.

9. Angle opposite the smaller given side, obtuse. No solution.

Any other example is reduced to one of the above by suitable trigonometric transformations.

References. In the following, the solutions are given as problems in descriptive geometry, and a knowledge of that subject is required to understand them.

A. E. Church, *Elements of Descriptive Geometry*, New York, 1864.

C. Margerie et E. Racine, *Traité de Géométrie Descriptive*, Paris, 1883.

C. A. Waldo, *Manual of Descriptive Geometry*, Boston, 1895.

F. Chomé, *Cours de Géométrie Descriptive de l'Ecole Militaire*, première partie, Paris, 1898.

W. S. Hall, *Descriptive Geometry,* New York, 1903.
S. E. Warren, *Elements of Descriptive Geometry,* New York, 1905.
J. B. Millar, *Elements of Descriptive Geometry,* New York, 1905.

QUESTIONS AND DISCUSSIONS.

Edited by W. A. Hurwitz, Cornell University, Ithaca, N. Y.

REPLIES.

34 [*1917,* 134, 341; *1920,* 114, 301, 405]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

I. Remarks by A. A. Bennett, Baltimore, Md.

For the functional equation of Simpson's Rule, the origin is a point specially characterized. If smoothness at the origin be not required, solutions which are continuous and contain an infinite number of parameters may be obtained which present irregularities at this point. In particular the following one-parameter family of analytic functions with an essential singularity at the origin satisfy the equation:

$$x^a \sin(b \log x - c),$$

where a and b are constants satisfying certain transcendental equations and where c is an arbitrary parameter. This solution may furthermore be added to any other solution to yield a solution.

For the equation

$$\int_0^h f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

if f is differentiable,[1] and $f(0) = 0$, becomes, after differentiation

$$6f(x) = [4f(x/2) + f(x)] + x[2f'(x/2) + f'(x)].$$

Testing $f(x) \equiv x^a \sin(b \log x - c)$ we have

$$f'(x) = x^{a-1}[a \sin(b \log x - c) + b \cos(b \log x - c)].$$

Substituting throughout this gives

$$\begin{aligned}6x^a \sin(b \log x - c) = (x^a/2^{a-2}) \sin(b \log x/2 - c) &+ x^a \sin(b \log x - c)\\ &+ (x^a/2^{a-2})[a \sin(b \log x/2 - c) + b \cos(b \log x/2 - c)]\\ &+ x^a[a \sin(b \log x - c) + b \cos(b \log x - c)].\end{aligned}$$

Collecting terms, and canceling x^a, this yields

$$\begin{aligned}(5 - a) \sin(b \log x - c) &- b \cos(b \log x - c)\\ &= (1/2^{a-2})[(a + 1) \sin(b \log x/2 - c) + b \cos(b \log x/2 - c)].\end{aligned}$$

Writing $b \log x/2 - c$ as $(b \log x - c) - b \log 2$ and expanding the terms in which this occurs, then equating coefficients of $\sin(b \log x - c)$ and of $\cos(b \log x - c)$, one obtains

$$\begin{cases}(5 - a)\cdot 2^{a-2} = (a + 1) \cos(b \log 2) + b \sin(b \log 2)\\ b\cdot 2^{a-2} = - b \cos(b \log 2) + (a + 1) \sin(b \log 2).\end{cases}$$

These equations are satisfied by an infinite set of values of a, b.[2]

---

[1] The assumption of differentiability is not essential. Direct substitution of the function $x^a \sin(b \log x - c)$ in the given equation leads to a pair of conditions on a, b equivalent to those derived in the text.

[2] See the remarks by the Editor, immediately following the present article.